An Engineering Approach to Hazard Rate Distribution Functions

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A method is presented for deriving an empirical function of component hazard rates from unit time and failure histories. The probability distribution of the mean hazard rate over a selected interval is derived, together with an excellent approximation which may be used for obtaining confidence statements or conducting simulation studies. The method has been reduced to practice, including computer programs for the analysis of data and graphical display of the results. The method is applied to an item of airborne electronic equipment and the results discussed.

Nomenclature

$\hat{X}$ = estimator of X
$E(X)$ = expected value of X
$F(X)$ = cumulative distribution function of X
$H(t)$ = sample hazard integral
H_k = increment of $H(t)$
$N(t)$ = sample size at unit age t
N_k = mean sample size over the kth interval
$C(\tau)$ = measure of hazard rate variation
$h(t)$ = unit hazard integral
h_k = increment of $h(t)$
r_k = number of failures occurring at the kth failure point
$r(t)$ = cumulative number of failures occurring in the sample
t = unit age
t_k = unit age at kth failure point
$\lambda(t)$ = unit hazard rate at age t
τ = unit age measured from beginning of useful life

Introduction

A REPETITIVE problem in reliability engineering is the estimation of component hazard rates from test data. The customary approach is to assume a time-dependent function of hazard rate with unknown parameters, derive estimates of the parameters from the test data, and then accept or reject the hypothesized function by some "goodness of fit" criterion. This is not a completely satisfactory solution to the problem for at least two reasons.

1) In many cases of practical interest, the test data are not generated to suit the desired analytical methods; rather, the data are gathered as fall out from some other activity, and the analyst is required to extract what information he can from it. For example, the data may include only isolated portions of the life cycles of each of a number of units. A case in point is the in-plant time and failure data recorded by the Lockheed-California Company on a government furnished equipment (GFE) item of airborne electronic equipment. Figure 1 shows the number of units under observation vs unit age as measured by an integral time meter on each unit. Figure 2 shows the cumulative number of units observed vs unit age at first observation. While the data span the unit age interval from 0 to 286 hr, do not include the total history of any single unit over this interval.

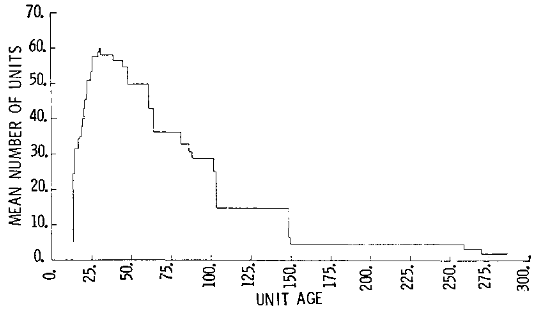

Fig. 1 Mean number of units under observation vs unit age.

2) In general, we may expect any hazard rate function to decrease as the infant mortality period draws to a close and begin increasing again as unit age enters the wearout period. In between, we expect either a constant hazard rate or at least a single inflection point. Significant variations from this general model may provide important clues to remediable causes of failure. The analysis of time and failure data in terms of an assumed hazard rate function tends to obscure these clues.

In searching for a more satisfactory solution to this problem, the Lockheed-California Company has developed a method for estimating the hazard rate function as an empirical function over the time domain covered by the test data. The method represents the instantaneous hazard rate as a time-dependent, empirical function of the average hazard rate over the useful life interval and provides a probability density function of the average hazard rate which can be used for the derivation of confidence statements, Monte Carlo simulations, etc. The method has been implemented by a computer program which performs all the arithmetical calculations and provides visual displays of the results.

The Hazard Rate Estimator

Consider a repairable unit with hazard rate $\lambda(t)$, which is an unknown function of unit age t. Suppose that we have a number of unit time and failure histories, and assume that each failure is independent of previous failures and repair actions. Letting $N(t)$ represent the number of units under observation at age t, we define a hazard integral $H(t)$ for the test sample by

$$H(t) = \int_0^t N(v)\lambda(v)dv \tag{1}$$

Failures in the test sample are homogeneous in H and hence are Poisson distributed.[1] Letting $r(t)$ represent the number of failures occurring in the interval $[0,t]$, then the expected

Received June 20, 1968; revision received March 5, 1969.
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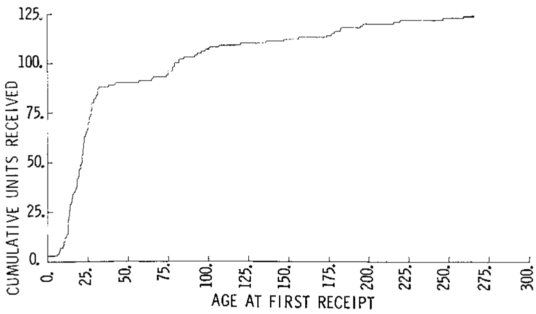

Fig. 2 Cumulative units received vs unit age at receipt.

value of $r(t)$ is equal to $H(t)$;

$$E[r(t)] = H(t) \tag{2}$$

For such processes, the observed value of $r(t)$ is a maximum likelihood estimator (Ref. 2, p. 167) for $H(t)$

$$\hat{H}(t) = r(t) \tag{3}$$

Let t_k, $k = 1, 2, \ldots$, represent the ages of failing units in the test sample at time of failure, ranked and indexed in order of increasing magnitude, and r_k represent the number of units failing at age t_k. Although theoretically no two units may fail at exactly the same age, limitations on meter reading accuracy may force $r_k > 1$.

The hazard integral $H(t_k)$ may be partitioned into increments H_j defined by

$$H_j = H(t_j) - H(t_{j-1}) \tag{4}$$

In any data set sufficient to define a hazard rate function, we assume the intervals $[t_{j-1}, t_j]$ will be sufficiently small so that $N(t)$ and $\lambda(t)$ will not both vary greatly over the interval. Under this assumption, we redefine the incremental hazard integral H_j in terms of the average number of units [/] N_j under observation over the interval. We define N_j by

$$N_j = (t_j - t_{j-1})^{-1} \int_{t_{j-1}}^{t_j} N(t)dt \tag{5}$$

Then H_j may be redefined under the assumption by

$$H_j = N_j \int_{t_{j-1}}^{t_j} \lambda(t)dt \tag{6}$$

Using this definition of H_j, then $H(t)$ at the point t_k is also redefined by

$$H(t_k) = \sum_{j=1}^{k} H_j \tag{7}$$

By the same arguments leading to Eq. (3), r_j is a maximum likelihood estimator for H_j,

$$\hat{H}_j = r_j \tag{8}$$

The hazard integral $H(t)$ is a parameter of the test sample, for which both $N(t)$ and $\lambda(t)$ may vary greatly over the interval $[0,t]$. What is desired is a corresponding parameter for the population which is free of this bias. We therefore define both an incremental and a cumulative unit hazard integral h_j and $h(t)$, respectively, by

$$h_j = \int_{t_{j-1}}^{t_j} \lambda(t)dt \tag{9}$$

$$h(t) = \int_0^t \lambda(v)dv \tag{10}$$

Similarly to Eq. (7), $h(t)$ at the point t_k is then defined by

$$h(t_k) = \sum_{j=1}^{k} h_j \tag{11}$$

From Eq. (6) and (9), it may be stated that

$$H_j = N_j h_j \tag{12}$$

Since H_j is the sum of N_j identical terms h_j, then a maximum likelihood estimator for h_j may be derived from Eq. (8)

$$\hat{h}_j = r_j/N_j \tag{13}$$

An estimator for $h(t)$ at the point t_k may be then stated as

$$\hat{h}(t_k) = \sum_{j=1}^{k} \hat{h}_j \tag{14}$$

While the estimator of Eq. (14) does not meet maximum likelihood criteria, it is still a good one. The points $[t_k, \hat{h}(t_k)]$ may be plotted and a smooth curve drawn through them to represent the hazard integral. The infant failure and wearout periods will display, respectively, decreasing and increasing slopes. A useful life interval $[b,w]$ can be defined on this function to optimize burning and overhaul schedules for the component. We then define a new unit age variable τ by

$$\tau = t - b \tag{15}$$

The failure times are then redefined on τ, and reindexed to eliminate the excluded infant failures. Finally, the population and sample parameters and their estimators are redefined on τ by obvious extensions and the useful life redefined on the interval $[0,\omega]$.

The mean hazard rate $\bar{\lambda}$, over the useful life interval, may be estimated by

$$\hat{\bar{\lambda}} = \hat{h}(\omega)/\omega \tag{16}$$

The instantaneous hazard rate $\lambda(\tau)$ may be estimated from the slope of the estimator function for $h(\tau)$. We define a new function $c(\tau)$ by

$$c(\tau) = \hat{\lambda}(\tau)/\hat{\bar{\lambda}} \tag{17}$$

The function $c(\tau)$ is the "best" estimator of the variability of the hazard rate, i.e., of the nature of the function $\lambda(\tau)$. Local anomalies in the function may provide clues to the causes of failure, as will be illustrated by an example in the sequel. The function $c(\tau)$ may also contribute to increased refinement of systems reliability analyses, etc. In the interests of brevity, we will consider its application only to recoverable systems for which mission duration is much shorter than useful component life.

For such systems, component hazard rates may be considered constant over the mission interval. It will be instructive,

Table 1 Comparison of convolution and gamma distributions[a]

① $F(h)$	② h^b	③ h^c	④ Relative error
0.01	0.997	0.817	−0.181
0.05	1.351	1.222	−0.095
0.10	1.581	1.489	−0.058
0.20	1.906	1.864	−0.022
0.30	2.177	2.172	−0.022
0.40	2.435	2.462	+0.011
0.50	2.704	2.755	+0.019
0.60	3.000	3.070	+0.023
0.70	3.351	3.433	+0.024
0.80	3.814	3.893	+0.021
0.90	4.558	4.594	+0.008
0.95	5.275	5.231	−0.008
0.99	6.903	6.570	−0.048

[a] $N_j = j$, $r_j = 1$, $j = 1, \ldots, 10$.
[b] $F(h) = [1 - \exp(-h)]^k$.
[c] Approximating gamma distribution.

however, to consider the probability distribution of component age during this mission interval. This is a consideration which is too often overlooked in reliability analyses based on hazard rates which vary with unit age. Unit age at the beginning of a mission may be nonuniformly distributed for a variety of reasons. In any case, we assume that its cumulative distribution function $F(\tau)$ is known.

The variable τ can be eliminated between the functions $F(\tau)$ and $c(\tau)$ by numerical methods to define $c(F)$ on the unit interval. In general, $c(F)$ is not monotonically increasing, so that it cannot be inverted to provide a distribution function $F(c)$. It can, however, be sampled on the unit interval for Monte Carlo simulations. A random value of the instantaneous hazard rate λ can then be derived as the product of a random value from $c(F)$ and a random value from the distribution of $\bar{\lambda}$, the mean hazard rate over the useful llfe. We consider now this distribution of $\bar{\lambda}$.

Distribution of Mean Estimator

The parameter $2r_jH_j/\hat{H}_j$ and hence also the parameter $2r_jh_j/\hat{h}_j$ has a chi-square distribution with $2r_j$ degrees of freedom (Ref. 2, pp. 165 and 194). This is a special case of the gamma distribution with mean $2r_j$ and variance $4r_j$. Since $\hat{h}_j$ is a constant, we may consider h_j in a purely mathematical sense as a random variable having a gamma distribution with mean and variance given by

$$E(h_j) = r_j/N_j \tag{18}$$

$$\text{var}(h_j) = r_j/N_j{}^2 \tag{19}$$

Under the hypothesis that the h_j are statistically independent the pdf of $h(\tau_k)$ can then be considered as the convolution of the pdf's of $h_j, j = 1, \ldots, k$. These convolutions can be performed by numerical methods on a computer.

The distribution $h(\tau_k)$ can also be quite closely approximated by less arduous means. If $N_j = N, j = 1, \ldots, k$, then the parameter $2r(\tau_k)h(\tau_k)/\hat{h}(\tau_k)$ has a chi-square distribution with $2r(\tau_k)$ degrees of freedom (Ref. 2, pp. 165 and 194). This is again a special case of the gamma distribution with mean $2r(\tau_k)$ and variance $4r(\tau_k)$. For this particular case, therefore

$$E[h(\tau_k)] = \sum_{j=1}^{k} E(h_j) \tag{20}$$

$$\text{var}[\text{h}(\tau_k)] = \sum_{j=1}^{k} \text{var}(h_j) \tag{21}$$

When N_j is not identical for each interval, we might still anticipate that $h(\tau_k)$ will approximate a gamma distribution with mean and variance as given by Eqs. (20) and (21).

For $N_j = j, r_j = 1, j = 1, \ldots, k$, it can be shown that the cumulative distribution function of $h(\tau_k)$ is given by

$$F(h) = [1 - \exp(-h)]^k \tag{22}$$

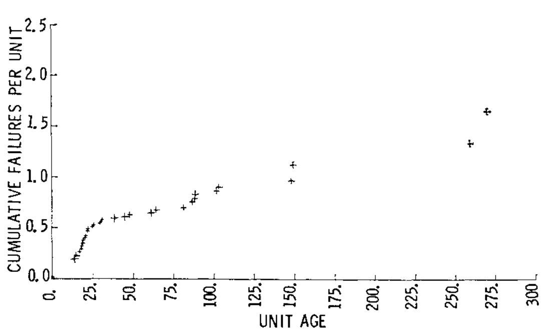

Fig. 3 Cumulative failures per unit vs unit age.

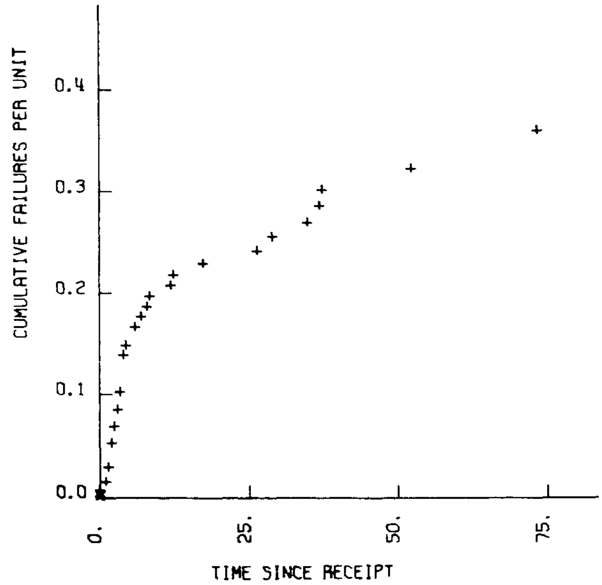

Fig. 4 Cumulative failures per unit vs time since receipt.

Column 2 of Table 1 gives the values of h for selected fractiles of F for the case in which $k = 10$. Column 3 of Table 1 shows the corresponding values of h from the gamma distribution with mean and variance as defined by Eqs. (20) and (21). Relative errors in this approximation are shown in column 4.

Whichever approach is taken to the distribution of $h(\omega)$, the distribution of $\bar{\lambda}$ is derived from it by linear transformation using

$$\bar{\lambda} = h(\omega)/\omega \tag{23}$$

An Example

In-plant time and failure data were collected on an item of military electronic equipment, installed in the P3 Orion as GFE. Each unit was functionally tested by the supplier before shipment. On receipt, the unit was again functionally tested in the Lockheed-California Company laboratory.

Units which failed the test were either repaired in the laboratory or returned to the supplier for repair at his facility. After passing the functional test, the items were installed in aircraft and operated during the production flight test cycle. Units which were reported to malfunction in the airplane were returned to the laboratory for retest and disposition.

Each unit bore a unique serial number and incorporated a small timer that recorded cumulative operating hours. A record of serial number and timer reading was made on each of the following occasions: 1) receipt by laboratory, 2) completion of functional test, 3) removal from aircraft for reported malfunction, 4) confirmation of malfunction in laboratory, and 5) delivery to customer.

Results and Discussion

All the data were processed with a time base of unit age to determine the mean number of units under observation over the intervals between failures, and the results plotted as shown in Fig. 1. This represents a total of 5130 hrs. of observation on 124 units. Data on these units were processed to determine the estimator points, $[t_k, \hat{h}(t_k)]$, for the hazard integral. The results are shown in Fig. 3. A straight line fitted to the points shown after 25 hr would have a slope corresponding to a 220 hr MTBF.

The infant failure phenomenon is clearly shown by the steeper slope in the first 25 hr of life. However, the upward concavity in this portion of the curve is anomalous. A second anomaly appears between 75 and 100 hr, where the cumulative

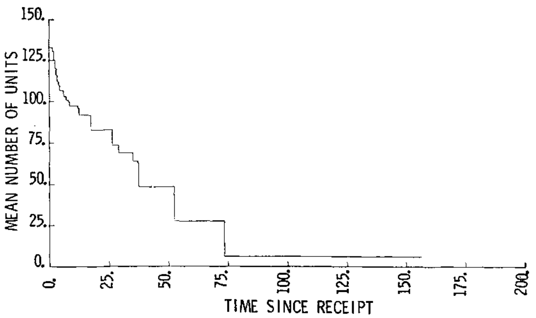

Fig. 5 Mean number of units under observation vs time since receipt.

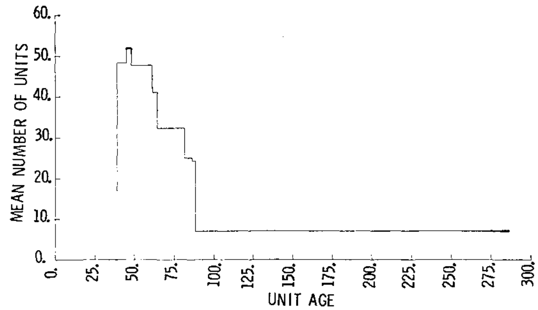

Fig. 7 Mean number of units under observation vs unit age—first 12.5 hr after receipt deleted.

failures per unit increase rather suddenly. Figure 2 shows the cumulative number of units received as a function of unit age at first receipt. This distribution might explain the anomalies in Fig. 3 if we assume that the malfunctions contributing to the points of Fig. 3 came from two sources: 1) the inherent hazard rate of the unit λ_1 considered as a function of cumulative operating hours and 2) an additional hazard rate increment λ_2 considered as a function of operating hours since receipt. Such an increment could arise from inadequate supplier final inspection or functional test, from damage during shipping, or from a variety of other causes.

To examine this second potential source of malfunction, the data were reanalyzed on the basis of time since receipt and the results plotted as shown in Fig. 4. Figure 5 shows the mean number of units under observation for the same data. Figure 4 shows a high hazard rate for the first few hours of observation after receipt, confirming the hypothesis of an additional hazard rate increment. This increment appears to have been filtered out in the first 12.5 hr of observation after receipt.

The first 12.5 hr after receipt were then considered as a burnin period and the data from this interval were discarded, leaving a total of 3700 hr of observation on 95 units. The remaining data were reanalyzed on the basis of unit age and the results plotted as shown in Figs. 6 and 7. The last malfunction occurred at a unit age of 88.5 hr, while the last observation was made at a unit age of 286.0 hr In the interval, a total of 1390.5 hr of failure free-operation was observed. The final point in Fig. 6 represents an assumed impending failure at the last instant of observation.

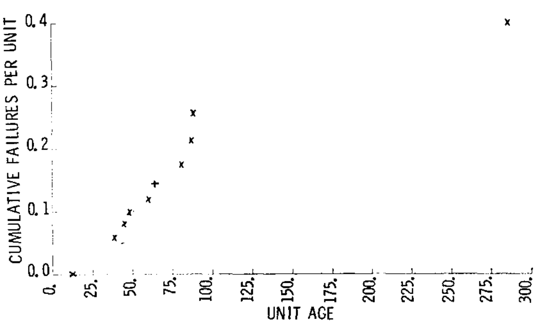

Fig. 6 Cumulative failures per unit vs unit age—first 12.5 hr after receipt deleted.

Subsequent to this, similar studies were made in house on three additional units, and identical phenomena were observed in all cases, i.e., 1) there was a high incidence of failure in the first few hours after receipt, which was independent of unit age, 2) when these early data were deleted, the infant reliability failure phenomenon was clearly visible, and 3) the mature reliability of the units as installed in the aircraft, after quality and infant reliability failures had been filtered out, was several times higher than the average values given by the raw data.

Conclusions

1) The cumulative number of failures per unit is a good estimator of the hazard integral, and with changes in the time base can be used to separate the effects of independent sources of malfunction.

2) The infant failure phenomenon was clearly demonstrated on the items of GFE studied.

3) The GFE items studied have a high mean time between failure (MTBF) in their mature installed state, but newly received units have a high early failure rate, independent of the age of the unit at receipt.

References

[1] Feller, W., *An Introduction to Probability Theory and Its Application*, 2nd ed., Wiley, New York, 1967, Chap. 17, p. 400.

[2] Lloyd, D. K. and Lipow, M., *Reliability: Management, Methods, and Mathematics*, Prentice-Hall, Englewood Cliffs, N. J., 1962, Chap. 7.